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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The Editors may also accept contributions from others.

Contributors to Part I will be presented 25 copies of reprints of their contributions. Extra copies will be supplied, if desired, at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary and Editor M. T. NARANIENGAR, M.A., 24, Mallesvaram, Bangalore, or Joint Editor K. ANANDA RAO, M.A., Professor, Presidency College, Madras.

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Enquiries by intending members of the Society and all other communications may be addressed to the Hony. Joint Secretary P. V. SESHU AIYAR, B.A., L.T., Principal, Government College, Kumbakonam.

The Indian Mathematical Society.

PROGRESS REPORT.

1. The following gentlemen have been admitted as Members of the Society at concession rate:—

(1) T. Vijayaraghavan, Research Scholar, Madras University, "Fieldhouse," Mambalam, Madras.

(2) M. Natarajan, B.A. (Hons.), Clerk, Military Test Audit Staff, c/o C. M. A., Poona Dt., Poona.

A Correction. It is regretted that in the last Progress Report the new Member's name from Lahore was wrongly given as Han Lall Chitkara. The correct name is Ram Lall Chitkara.

2. In recognition of the valuable services rendered to the Society by Mr. V. Ramaswami Iyer, Founder of the Society, and on the occasion of the Society attaining its *majority* (the Society was started in 1907), the Managing Committee has been pleased to make him an Honorary Member of our Society and has resolved to present him with a suitable address.

3. Mr. G. A. Srinivasan, the Asst. Secretary, having left Madras, Mr. A. Narasinga Rao, Asst. Professor of Mathematics, Presidency College, has been appointed Asst. Secretary in his place and Mr. G. A. Srinivasan has been taken as an Associate in the place of Mr. A. Narasinga Rao, and the Managing Committee has passed a resolution placing on record its grateful appreciation of the valuable services rendered by Mr. G. A. Srinivasan as Asst. Secretary of the Society for over a year.

4. It is proposed to hold the next Conference of the Society at Bangalore somewhere about the end of this year or the beginning of next year, and members are hereby requested to try to attend the Conference in large numbers and present papers to the Conference so as to make it a great success. The detailed programme, etc., will be communicated in due course.

KUMBAKONAM,
25th July 1925

P. V. SESHU AIYAR,
Hony. Secretary.

The Indian Mathematical Society.

ANNOUNCEMENT.

A member of the Hindustan Association of America has donated a Gold Medal known as "Mahamaya Gold Medal" to be awarded to the author of the best original essay on "Hindu-Arabic Numerals," and the Association has requested the Calcutta Mathematical Society to conduct the work of collecting the essays and selecting the best one among them by a Committee appointed for the purpose. The President of the Calcutta Mathematical Society, Mr. Ganesh Prasad, has constituted the following Committee of 5 members for awarding the medal :—

1. The President of the Calcutta Mathematical Society (*ex-officio* Chairman).

2 & 3. Two Members of the Calcutta Mathematical Society to be elected by that Society.

4. One Member of the Indian Mathematical Society to be elected by that Society.

5. One Member of the Benares Mathematical Society to be elected by that Society.

Competitors for the medal must be graduates or teachers in any Indian University. The object of the essay shall be to make a critical study of the theories advanced by various prominent Mathematicians, one group maintaining the Arabic origin of the numerals and the other group upholding their Hindu origin. The essays should reach the Secretary of the Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta, on or before the 31st October of 1925, in three type-written copies.

KUMBAKONAM,
25th July 1925.

P. V. SESHU AIYAR,
Hony. Secretary.

For
$$\bar{L} = \frac{\Delta K}{\pi \bar{\Delta}^2} \tag{18.10}$$

and the constant L we considered before was given by
$$L = \frac{K}{\pi \Delta} \qquad \text{(equation 9.3)}$$

or in the present notation $\frac{K}{\pi \bar{\Delta}}$

so that
$$\bar{L} = \frac{\Delta}{\bar{\Delta}} L = \frac{\Delta}{\bar{\Delta}} L_{00} \tag{9.4}$$
$$= -\frac{\Delta}{\bar{\Delta}} \tag{10.6}$$

since $\alpha = 0$ in this case.

Let us note that this is true because K does not depend on the choice of the cylinder which we take for figure of comparison.

§ 20. Let us now change our definition of α also slightly. Instead of defining it by
$$\alpha = \frac{1}{\pi} \int \bar{H}\, \bar{\zeta} \cos m\psi \, d\psi$$

let us take
$$\alpha = \frac{1}{\pi} \int \bar{H}\, \zeta_0 \cos m\psi \, d\psi. \tag{20.1}$$

This will not alter the equations (19.2), (19.3). For it is sufficient for these formulæ that the expression (20.1) for α should vanish with the integral $\int \bar{H}\, \bar{\zeta} \cos m\psi \, d\psi$.

But if
$$\int \bar{H}\, \bar{\zeta} \cos m\psi \, d\psi = 0$$

the figure F reduces to the Elliptic Cylinder E and therefore ζ_0 becomes ζ_{00}.

Hence
$$\int \bar{H}\, \zeta_0 \cos m\psi \, d\psi = \int \bar{H}\, \zeta_{00} \cos m\psi \, d\psi$$
$$= \int \bar{H} \cos m\psi \left[\delta\rho + \frac{\cos^2\psi}{\bar{H}} (\rho + \delta\rho)\, \delta_q \right] d\psi \text{ by (18.6)}$$

Hence if $m > 2$, we have then
$$\int \bar{H}\, \zeta_0 \cos m\psi \, d\psi = 0.$$

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Thus, although we have changed our definition of α, the equations (19.2), (19.3) will remain unaltered.

The fact is that the ζ_{0s} and the L_{0s} are definite functions of ρ and therefore of Ω and in no way depend on the definition of α.

To prove this we note that the fundamental equation

$$\frac{2ab}{(a+b)^2} H \zeta_0 + \frac{1}{2\pi}\int \log R^2(\rho\rho)\, H'\zeta'_0\, d\psi' = \frac{\Delta}{\pi} W + \text{constant} \quad (20.2)$$

must be satisfied by all figures of equilibrium near the given cylinder E_0. Consequently the elliptic cylinder corresponding to the angular velocity given by $\Omega = \Omega_0 + \eta$ must also be such that when its equation with respect to E_0 as figure of comparison is

$$x = \sqrt{\rho + q + \zeta_0}\,\cos\psi\,;\quad y = \sqrt{\rho + \zeta_0}\,\sin\psi$$

the ζ_0 for it, is given by equation (20.2).

But we know from *a priori* considerations that, in this case, ζ_0 is given by the formula

$$\rho + \zeta_0 = f_1(\Omega_0 + \eta) \quad (20.3)$$

where

$$\rho = f_1(\Omega_0)$$

is the equation of the given elliptic cylinder. (20.4)

Hence

$$\zeta_0 = \frac{d\rho}{d\Omega}\eta + \frac{1}{2!}\frac{d^2\rho}{d\Omega^2}\eta^2 + \ldots\ldots + \frac{1}{s!}\frac{d^s\rho}{d\Omega^s}\eta^s + \ldots\ldots$$

But comparing this with the general value of ζ_0, *viz.*

$$\zeta_0 = \sum_{r+s>0} \zeta_{rs}\,\alpha^r\eta^s \quad (18.11)$$

we see that for Elliptical figures of equilibrium

$$\alpha = 0 \quad (20.5)$$

Moreover, since the co-efficients ζ_{rs} are independent of α, η, but are definite functions depending only on ρ, ψ, they must be the same for the same ψ, both for the elliptical figures as well as the non-elliptical figures of equilibrium. Thus we see that for the ζ_{0s} we can take constant values given by

$$\zeta_{0s} = \frac{1}{s!}\left(\frac{d^s\rho}{d\Omega^s}\right)_{\Omega=\Omega_0} \quad (20.6)$$

In exactly the same way, we see that if

$$\frac{T_m}{\Delta} = f_2(\Omega_0)$$

we get

$$\frac{\overline{T}_m}{\Delta} = f_2(\Omega_0 + \eta).$$

From (19.4) therefore

$$\overline{L} = -\Delta . \frac{\overline{T}_m}{\Delta} = -\Delta \left[\frac{T_m}{\Delta} + \sum_{s=1}^{\infty} \frac{\eta^s}{s!} \left\{ \frac{d^s}{d\Omega^s} \left(\frac{T_m}{\Delta} \right) \right\} \right]_{\Omega = \Omega_0}.$$

or $$L_{0s} = -\frac{\Delta}{s!} \left[\frac{d^s}{d\Omega^s} \left(\frac{T_m}{\Delta} \right) \right]_{\Omega = \Omega_0}. \qquad (20.7)$$

From (20.7) we deduce the following equation—

If the given elliptic cylinder E_0 is defined by the equation

$$T_m = 0$$

then $$L_{00} = 0 \text{ and } L_{01} = -\frac{d\,T_m}{d\Omega_0}. \qquad (20.8)$$

§ 21. So far we have considered the ζ_{rs} and L_{rs} for which $r = 0$. Let us now see how we can calculate the ζ_{rs} and the L_{rs} for which $r \neq 0$.

The process is exactly similar to the one by which we calculated

$$\zeta_n,\ L_{n-1}.$$

The calculation of the ζ_{rs} and the L_{rs} does not give rise to any difficulty. If we use the condition that the volume of the liquid is constant (given by the equation (9.4) in which we must put ζ_0 in the place of ζ) the expressions for the ζ_{rs} can be uniquely determined.

We now proceed to show that any functions ζ_{rs} can be calculated without any ambiguity when we know all the ζ_{ij} for which

$$i \leqslant r;\ j \leqslant s;\ i + j < r + s. \qquad (21.1)$$

Let us first show that under these conditions we can at once calculate

$$L_{r-1,\,s}$$

Equation (18.9) gives

$$\frac{2ab}{(a+b)^2} \mathrm{H}\zeta_0 + \frac{1}{2\pi}\int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}\zeta'_0\, d\psi'$$

$$= \frac{\Delta}{\pi} \mathrm{W} - \bar{\mathrm{L}}\,\bar{\mathrm{H}}\,(\zeta_0 - \zeta_{00}) + \text{constant}.$$

From our new definition of α

$$\bar{\mathrm{L}}\,\alpha = \frac{\bar{\mathrm{L}}}{\pi}\int \bar{\mathrm{H}}\zeta_0 \cos m\psi\, d\psi$$

$$= \frac{\Delta}{\pi^2}\int \mathrm{W}\cos m\psi\, d\psi - \frac{\mathrm{T}_m}{\pi}\int \mathrm{H}\zeta_0 \cos m\psi\, d\psi. \quad (21\ 2)$$

Substitute in this the expansion (18.11) for ζ_0, *viz.*

$$\zeta_0 = \sum_{r+s>0} \zeta_{rs}\, \alpha^r\, \eta^s.$$

From this we can deduce an expression for $\bar{\mathrm{L}}$ of the form we took for it, *viz.*

$$\bar{\mathrm{L}} = \mathrm{L}_{00} + \sum_{r+s>0} \mathrm{L}_{rs}\, \alpha^r\, \eta^s.$$

For the right-hand side of equation (21.2) contains only terms which vanish with α.

In fact the terms not depending on α reduce to

$$\frac{\Delta}{\pi^2}\int \mathrm{W}_0 \cos m\psi\, d\psi - \frac{\mathrm{T}_m}{\pi}\int \mathrm{H}\zeta_0 \cos m\psi\, d\psi$$

where W_0 is what W becomes when $\alpha = 0$ or we replace ζ_0 by ζ_{00}.

But ζ_{00} satisfies the equation

$$\frac{2ab}{(a+b)^2} \mathrm{H}\zeta_{00} + \frac{1}{2\pi}\int \log \mathrm{R}^2(\rho\rho)\, \mathrm{H}'\zeta'_{00}\, d\psi' = \frac{\Delta}{\pi} \mathrm{W}_0 + \text{constant},$$

since the cylinder E is a figure of equilibrium.

Therefore multiplying by $\cos m\psi\, d\psi$ and integrating, we get

$$\frac{\Delta}{\pi}\int \mathrm{W}_0 \cos m\psi\, d\psi - \mathrm{T}_m \int \mathrm{H}\zeta_0 \cos m\psi\, d\psi = 0.$$

Thus, if the expansion of W be

$$\mathrm{W} = \sum \mathrm{W}_{rs}\, \alpha^r\, \eta^s, \quad (21.3)$$

we have from (21.2) on equating the co-efficients of $\alpha^r \eta^s$ on both sides

$$L_{r-1,s} = \frac{\Delta}{\pi^2} \int W_{rs} \cos m\psi \, d\psi - \frac{T_m}{\pi} \int H \zeta_{rs} \cos m\psi \, d\psi. \quad (21.4)$$

But the right-hand side of this depends only on the ζ_{ij} for which (**21.1**) holds. For, consider the second term on the right-hand side.

If in the formula

$$\alpha = \frac{1}{\pi} \int \bar{H} \zeta_0 \cos m\psi \, d\psi$$

we replace $\bar{H}$ by its expansion

$$\bar{H} = H + H_1 \eta + H_2 \eta^2 + \ldots\ldots\ldots\ldots$$

and put in the value of ζ_0 given by (18.11), and equate co-efficients of the same powers of α, η, we get

$$\frac{1}{\pi} \int H \zeta_{10} \cos m\psi \, d\psi = 1$$

or

$$H \zeta_{10} = \cos m\psi \quad (21.5)$$

also since $0 = \frac{1}{\pi} \int \left\{ \begin{array}{l} \{ H_1 \eta + \ldots \} \{ \sum\limits_{r+s>0} \zeta_{rs} \alpha^r \eta^s \} \\ \quad + H \{ \zeta_{01} \eta \} + \sum\limits_{r+s>1} \zeta_{rs} \alpha^r \eta^s \end{array} \right\} \cos m\psi \, d\psi$,

equating co-efficients of $\alpha^r \eta^s$ on both sides, we get

$$\frac{1}{\pi} \int H \zeta_{rs} \cos m\psi \, d\psi = - \sum_{j=0}^{s-1} \frac{1}{\pi} \int H_{s-j} \zeta_{rj} \cos m\psi \, d\psi, \quad (21.6)$$

where the summation is to be replaced by 0 when $s = 0$.

We thus see that, if we know all the ζ_{ij} for which

$$i \leqslant r; \; j \leqslant s; \; i + j < r + s,$$

we know all the L_{ij} for which

$$i \leqslant r - 1; \; j \leqslant s.$$

Again, returning to the equation (18.9), if we put in it

$$\bar{L} = L_{00} + \sum_{i'+j'>0} L_{i'j'} \alpha^{i'} \eta^{j'}$$

$$\bar{H} = H + \sum_{h>0} H_h \eta^h$$

$$\zeta_0 - \zeta_{00} = \sum_{i>0} \zeta_{ij} \alpha^i \eta^j$$

and equate co-efficients of $\alpha^r \eta^s$ on both sides we get

$$\frac{2ab}{(a+b)^2} H\zeta_{rs} + \frac{1}{2\pi}\int \log R^2(\rho\rho)\, H'\zeta'_{rs}\, d\psi' \tag{21.7}$$

$$= \frac{\Delta}{\pi} W_{rs} - \Sigma L_{i'j'} H_h \zeta_{ij} + \text{constant},$$

where the summation is extended to all the values of i, j, i', j', h satisfying the equations

$$i + i' = r;\ j + j' + h = s;\ i > 0.$$

This sum will thus contain a term depending on ζ_{rs}, *viz.*

$$L_{00} H\zeta_{rs} \equiv -T_m H\zeta_{rs} \text{ (from 20.7)}$$

and, excepting this, all the other ζ_{ij} will satisfy the relations (21.1) *i.e.*

$$i \leqslant r;\ j \leqslant s;\ i + j < r + s.$$

Moreover, for the $L_{i'j'}$ we have $i' \leqslant r-1$; $j' \leqslant s$ since in this sum $i>0$.

Consequently, if we transpose the term in ζ_{rs} to the left-hand side, we shall get

$$\left\{\frac{2ab}{(a+b)^2} - T_m\right\} H\zeta_{rs} + \frac{1}{2\pi}\int \log R^2(\rho\rho)\, H'\zeta'_{rs}\, d\psi'$$

$$= Z + \text{constant}, \tag{21.8}$$

where Z will be a known function when we know all the ζ_{ij} which satisfy (21.1).

Thus $$\int H\zeta_{rs} \cos l\psi\, d\psi = \frac{1}{T_l - T_m}\int Z \cos l\psi\, d\psi$$

$$\int H\zeta_{rs} \sin l\psi\, d\psi = \frac{1}{T'_l - T_m}\int Z \sin l\psi\, d\psi$$

excepting in the cases $l = 0$ and $l = m$ for the first integral. As regards these two cases the integral in question can be calculated by using the condition that the volume of the liquid is constant (9.4), or by means of the equation (21.6).

Thus we know all the integrals of this kind and therefore the function $H\zeta_{rs}$ is completely known without any ambiguity, being defined by a Fourier Series which is, as will be shown later on, absolutely and uniformly convergent.

Let us note, however, that the functions defined are really particular integrals of the type discussed in § 13.

We have thus shown how we can, by properly defining the adjacent surface, pass from the given elliptic cylinder to the adjacent surface uniquely, and without any ambiguity.

Let us note that when the adjacent surface F is one of equilibrium, we must have

$$K = 0 \text{ or } L = 0 \text{ or } \bar{L} = 0$$

and equation (21.2) gives for the Elliptic Cylinder defined by $T_m = 0$

$$\bar{L}\,\alpha = \frac{\Delta}{\pi^2} \int W \cos m\psi \, d\psi$$

and therefore $L \equiv 0$ is the same as $\int W \cos m\psi \, d\psi = 0$, or $A = 0$ as in the other method.

PART III.

The Expansions for the Moment of Inertia and the Angular Momentum.

§ 22 So far we have only considered the first part of our problem, namely, the existence of the non-elliptical figures of equilibrium close to a given elliptic cylinder. We have shown how we can pass from the given cylinder to an adjacent surface, not necessarily one of equilibrium and that this can be done uniquely, if we define the adjacent surface properly, and if we keep the volume of the liquid constant. We took two different arbitrary relations to be satisfied by the surface potential of the new surface, *viz*.

$$U + (\Omega_0 + \eta)(x^2 + y^2) \equiv K \left(\frac{x^2}{\rho + q + \delta\rho + \delta_q} + \frac{y^2}{\rho + \delta\rho} \right) + \text{constant},$$

and
$$U + (\Omega_0 + \eta)(x^2 + y^2) \equiv AY + \text{constant}.$$

We have shown how corresponding to each of these there is but one unique surface which, if defined by the equations

$$x = \sqrt{\rho + q + \zeta} \cos\psi \,;\; y = \sqrt{\rho + \zeta} \sin\psi,$$

is such that under certain conditions* ζ can be expressed as an

---

* The conditions are that

$$|\zeta_n| < l_n \text{ and } \Sigma\, l_n \alpha^n$$

is convergent for small enough α or, in the case of two parameters,

$$|\zeta_{rs}| < l_{rs} \text{ and } \Sigma\, l_{rs}\, \alpha^r \eta^s \text{ is convergent.}$$

These will be discussed in the last part dealing with the convergence of the series.

absolutely and uniformly convergent power series. We have shown moreover that if the adjacent surface F is to be one of equilibrium we need only take K = 0 in the one case, and A = 0 in the other, and that when F is an equilibrium figure the two equations

$$K = 0 \text{ or } L = 0 \text{ and } A = 0$$

are identical at the point of bifurcation given by $T_m = 0$. The question now arises whether these new figures of equilibrium are stable or unstable. For this purpose we have to investigate whether the angular momentum of the new figure F is greater or less than that of the given cylinder at the point of Bifurcation. In fact we shall show that if the angular momentum for the new figure is greater than that for the given cylinder there is a minimum of energy, and hence, by the principle of Thomson and Tait, the new figure will be stable*.

Let us therefore proceed to find the angular momentum of the figure F. For this purpose let us use the second definition of the adjacent surface namely that on it we must have

$$U + (\Omega_0 + \eta)(x^2 + y^2) = A \cos m\psi + \text{constant},$$

when the given cylinder at the point of Bifurcation is given by

$$T_m = 0.$$

After we have found the expression for the angular momentum for this surface (which so far we have not taken as one of equilibrium) we shall put in the condition of equilibrium, *viz.*

$$A = 0.$$

This will then give us the angular momentum of the new figure of equilibrium close to the given cylinder. As we have shown that

$$L = 0 \text{ and } A = 0$$

are one and the same condition the fact that we shall use the second form of the condition satisfied by the adjacent surface to find the angular momentum, cannot make any difference. We could of course equally well start with the first form of condition, but it will involve a little more unnecessary labour.

§ 23. But, before we proceed to get the expansions for the Moment of Inertia and the Angular Momentum for the figure F, let us find the expansion of

$$A \equiv \frac{1}{\pi} \int W \cos m\psi \, d\psi.$$

---

* See the note 2 on p. 6 in the Introduction.

From equation (18.5) we have

$$W = \eta\,(\rho + q\cos^2\psi + \zeta) + U_2 + U_3 + \ldots\ldots\ldots\ldots$$

where by (7.3)

$$U_n = \operatorname*{Lim}_{\substack{v=\rho \\ u\to\rho-0}} \left\{ \begin{array}{l} \dfrac{1}{n!}\dfrac{\partial^{n-1}}{\partial v^{n-1}}\displaystyle\int K\zeta'^n\,d\psi' + \dfrac{\zeta}{(n-1)!\,1!}\dfrac{\partial^{n-1}}{\partial v^{n-2}\,\partial u}\times \\ \qquad\qquad\qquad\qquad \displaystyle\int K\zeta'^{n-1}\,d\psi' \\ + \ \ldots\ + \dfrac{\zeta^r}{(n-r)!\,r!}\dfrac{\partial^{n-1}}{\partial u^r\,\partial v^{n-r-1}}\times \\ \qquad\qquad\qquad\qquad \displaystyle\int K\zeta'^{n-r}\,d\psi' + \ldots\ldots \\ + \ \ldots\ + \dfrac{\zeta^{n-1}}{(n-1)!\,1!}\dfrac{\partial^{n-1}}{\partial u^{n-1}}\displaystyle\int K\zeta'\,d\psi'. \end{array} \right\}$$

Thus we have

$$W = \left| \begin{array}{l} \eta\,(\rho + q\cos^2\psi + \zeta) \\ + \operatorname{Lim}\left[\dfrac{1}{2}\dfrac{\partial}{\partial v}\displaystyle\int K\zeta'^2\,d\psi' + \zeta\,\dfrac{\partial}{\partial u}\displaystyle\int K\zeta'\,d\psi'\right] \\ + \operatorname{Lim}\left[\dfrac{1}{6}\dfrac{\partial^2}{\partial v^2}\displaystyle\int K\zeta'^3\,d\psi' + \dfrac{\zeta}{2}\dfrac{\partial^2}{\partial u\,\partial v}\displaystyle\int K\zeta'^2\,d\psi' \right. \\ \qquad\qquad\qquad \left. + \dfrac{\zeta^2}{2}\dfrac{\partial^2}{\partial u^2}\displaystyle\int K\zeta'\,d\psi'\right] \\ + \ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ + \operatorname{Lim}\left[\dfrac{1}{n!}\dfrac{\partial^{n-1}}{\partial v^{n-1}}\displaystyle\int K\zeta'^n\,d\psi' + \ldots\ \ldots \right. \\ \qquad\qquad \left. + \dfrac{\zeta^{n-1}}{(n-1)!}\dfrac{\partial^{n-1}}{\partial u^{n-1}}\displaystyle\int K\zeta'\,d\psi'\right] \\ + \ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \end{array} \right| \tag{23.1}$$

In this let us put

$$\zeta = \sum_{r+s>0} \zeta_{rs}\,\alpha^r\,\eta^s \tag{23.2}$$

where the ζ_{rs} are functions of ψ and arrange W as a power series in α, η say

$$W = \Sigma\, W_{rs}\,\alpha^r\,\eta^s. \tag{23.3}$$

Then we get

$$A = \frac{1}{\pi} \int W \cos m\psi \, d\psi = \frac{1}{\pi} \Sigma \, \alpha^r \eta^s \int W_{rs} \cos m\psi \, d\psi.^*$$

Now, it is obvious from our expression for W given by (23.1) that, when $m > 2$, the terms in the expansion of A will be at least of the second order with respect to α, η : for, U_n is of order n in ζ, ζ' and, when $m > 2$, we have

$$\int (p + q \cos^2 \psi) \cos m\psi \, d\psi = 0.$$

The expansion for A will be therefore of the form

$$A \equiv A_2 \alpha^2 + A_3 \alpha^3 + \ldots\ldots + (B + \Sigma) \alpha \eta + C_2 \eta^2 + C_3 \eta^3 + \ldots\ldots \tag{23.4}$$

where A_2, A_3 ...... B, C_2, C_3 ......

are fixed numbers (depending of course on rather complicated integrations), and Σ is a power series in α, η which vanishes for $\alpha = 0 = \eta$.

Further, we note that $C_2 \equiv 0$; for, we have

$$C_2 = \text{co-efficient of } \eta^2 \text{ in } \frac{1}{\pi} \int_0^{2\pi} W \cos m\psi \, d\psi$$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos m\psi \, d\psi \left[\zeta_{01} + \lim \left\{ \frac{1}{2} \frac{\partial}{\partial v} \int K \zeta'^2_{01} \, d\psi' + \zeta_{01} \frac{\partial}{\partial u} \int K \zeta'_{01} \, d\psi' \right] \right.$$

But, by (20.6), the ζ_{0s} are constants depending only on ρ;

$$\therefore \ \pi C_2 = \zeta^2_{01} \int_0^{2\pi} \cos m\psi \left[\lim \left(\frac{1}{2} \frac{\partial}{\partial v} + \frac{\partial}{\partial u} \right) \int K \, d\psi' \right] d\psi.$$

But by (6.2)

$$\iint K \, dv \, d\psi' = \text{potential of the Cylinder } v \text{ at the point } u$$

$$= \text{constant} - \frac{2\pi (bx^2 + ay^2)}{a + b}$$

$$= \text{constant} - \frac{2\pi ab (a \cos^2 \psi + b \sin^2 \psi)}{a + b}.$$

---

* This integration term by term is legitimate because of our assumptions with respect to ζ_{rs} which make the series (23.3) absolutely and uniformly convergent for all values of ψ.

$$\therefore \quad \pi C_2 = \zeta^2_{01} \lim \left(\frac{1}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial u}\right) \frac{\partial}{\partial v} \int_0^{2\pi} \cos m\psi \, d\psi$$

$$\left[\text{constant} - \frac{2\pi ab(a\cos^2\psi + b\sin^2\psi)}{a+b}\right]$$

$$\equiv 0, \text{ for } m > 2.$$

We shall show that for finding the first two terms of the angular momentum of the figure F, we need only calculate the number B and the number A_2 when m is even, and the number A_3 when m is odd; (for, when m is odd, $A_2 \equiv 0$.

§ 24. The Moment of Inertia.

Let ζ_0 be the moment of Inertia of the given cylinder. Then if

$$G(\rho, \psi) \equiv \frac{H(\rho, \psi)}{\Delta(\rho)} = \frac{\rho + q\sin^2\psi}{\sqrt{\rho(\rho+q)}},$$

the moment of inertia of the figure F defined by

$$x = \sqrt{\rho + q + \zeta}\cos\psi; \; y = \sqrt{\rho + \zeta}\sin\psi$$

is $S = \int (x^2 + y^2)\, dv$

$$= S_0 + \tfrac{1}{2}\int_0^{2\pi} d\psi \int_0^{\zeta} (\rho + q\cos^2\psi + u)\, G(\rho + u, \psi)\, du, \quad (24.1)$$

where v denotes the volume of the distorted cylinder.

and $$dv = \frac{\partial(x, y)}{\partial(u, \psi)}\, du\, d\psi = \tfrac{1}{2} G(\rho + u, \psi)\, du\, d\psi.$$

Let us find the first term in the expansion of $2(S - S_0)$. It is evidently equal to

$$\int G_0\, \zeta(\rho + q\cos^2\psi)\, d\psi$$

where $$G_0 = G(\rho\,\psi) = \frac{H(\rho\,\psi)}{\Delta(\rho)}.$$

Now $\zeta = \zeta_{10}\,\alpha + \zeta_{01}\,\eta + \zeta_{20}\,\alpha^2 + \zeta_{11}\,\alpha\eta + \zeta_{02}\,\eta^2 + \ldots\ldots;$

also from (21.5), $H(\rho\,\psi)\,\zeta_{10} = \cos m\psi$, and $m > 2$.

Therefore, we get only one term of the first degree in $2(S - S_0)$, *viz.*

$$\eta \int G_0\, \zeta_{01}(\rho + q\cos^2\psi)\, d\psi.$$

This is never equal to zero for it is equal to 2 $S'_0\eta$, where

$$S'_0 = \text{the value of } \frac{dS}{d\Omega} \text{ for } \Omega = \Omega_0:$$

for, by the fundamental equation* which must be satisfied by any surface of equilibrium (and therefore for all elliptic cylinders also), we have near the given cylinder

$$\frac{2ab}{(a+b)^2} G_0 \zeta + \frac{1}{2\pi} \int \log R^2 (\rho\rho) G'_0 \zeta' \, d\psi' = \frac{1}{\pi} W + \text{const.}$$

on putting

$$G_0 = \frac{H}{\Delta},$$

and the condition that the volume is constant is

$$\int d\psi \int_0^{\zeta} G(\rho + u, \psi)\, du = 0.$$

Since $\Omega = \Omega_0 + \eta$, and therefore $\frac{\partial}{\partial\Omega} = \frac{\partial}{\partial\eta}$,

$$\frac{dS}{d\Omega} = \tfrac{1}{2} \int_0^{2\pi} (\rho + q\cos^2\psi + \zeta)\, G(\rho + \zeta, \psi) \frac{\partial\zeta}{\partial\eta}\, d\psi$$

$$\frac{2ab}{(a+b)^2} G_0 \frac{\partial\zeta}{\partial\eta} + \frac{1}{2\pi} \int \log R^2 (\rho\rho) G'_0 \frac{\partial\zeta'}{\partial\eta} d\psi' = \frac{1}{\pi} \frac{\partial W}{\partial\eta} + \text{constant}$$

and

$$\int G_{,}(\rho + \zeta, \psi) \frac{\partial\zeta}{\partial\eta} d\psi = 0.$$

Making $\Omega = \Omega_0$ and consequently $\eta = 0$, $\zeta = 0$, we get noting that then

$$\frac{dS}{d\Omega} = S'_0, \quad \frac{\partial\zeta}{\partial\eta} = \zeta_{01},$$

$$S'_0 = \tfrac{1}{2} \int_0^{2\pi} (\rho + q\cos^2\psi)\, G_0 \zeta_{01}\, d\psi; \quad \int G_0 \zeta_{01}\, d\psi = 0.†$$

Hence if we put

$$S = S_0 + \sum_{r+s>0} S_{rs}\, \alpha^r \eta^s,$$

we see that

$$S_{10} = 0; \quad S_{01} = S'_0. \tag{24.2}$$

---

* Equation (17.2) wherein we put A = 0.

† This last equation is obvious from the fact that the ζ_{rs} are constants given by (20.6)

For all cases where m is even, this is enough for the investigation of stability. But for m odd, we shall see that η is of the order α^2 and therefore we require S_{20} also.

§ 25. **To find the value of ζ_{20}.**

For this purpose, we use the differential of the mutual potential energy for the figure F. Consider the mutual potential energy for the figure F. It is

$$V = -\tfrac{1}{2}\int\int \log(\text{distance})^2\, dm\, dm' \text{ for two dimensions}$$
$$= \tfrac{1}{2}\int\int \phi\, dm\, dm' \text{ for three dimensions,} \tag{25.1}$$

where the integrations are to be taken throughout the figure F. Let us use as before

$$x = \sqrt{\rho + q + u}\cos\psi,\quad y = \sqrt{\rho + u}\sin\psi$$

for any point inside or on this surface.

Then, if U (u) represents the potential at the point

$$x = \sqrt{\rho + q + u}\cos\psi,\quad y = \sqrt{\rho + u}\sin\psi$$

due to the whole of the figure F, we have clearly

$$V = \tfrac{1}{2}\int dT\, U(u) = \tfrac{1}{4}\int_0^{2\pi} d\psi \int_{-\rho}^{\zeta} U(u)\, G(\rho + u, \psi)\, du$$

$$= \tfrac{1}{4}\int_0^{2\pi} d\psi \int_{-\rho}^{\zeta} U(u)\, G(u)\, du, \text{ say.} \tag{25.2}$$

Let us now consider the differential of this with respect to a variation of the parameters α, η so that we have

$$\delta\zeta = \frac{\partial\zeta}{\partial\alpha}\delta\alpha + \frac{\partial\zeta}{\partial\eta}\delta\eta.$$

We have from (25.2)

$$\delta V = \tfrac{1}{4}\int_0^{2\pi} d\psi \int_{-\rho}^{\zeta} \delta U(u)\, G(u)\, du + \tfrac{1}{4}\int_0^{2\pi} U(\zeta)\, G(\zeta)\, \delta\zeta\, d\psi.$$

But we have

$$U(u) = -\int \log(\text{distance})^2\, dT'$$

$$= -\tfrac{1}{2}\int_0^{2\pi} d\psi' \int_{-\rho}^{\zeta'} \log r^2(u, u')\, G'(u')\, du'$$

where $r^2(u, u')$ = square of the distance of the two points denoted by

$$[\sqrt{\rho+q+u}\cos\psi, \sqrt{\rho+u}\sin\psi]\,;\ [\sqrt{\rho+q+u'}\cos\psi', \sqrt{\rho+u'}\sin\psi']$$

Hence the first term in the expression for δV is

$$\tfrac{1}{4}\int d\psi \int_{-\rho}^{\zeta} G(u)\,du \times -\tfrac{1}{2}\int \log r^2(u, \zeta')\, G'(\zeta')\,\delta\zeta'\,d\psi'$$

$$= \tfrac{1}{4}\int G'(\zeta')\,\delta\zeta'\,d\psi' \times -\tfrac{1}{2}\int d\psi' \int_{-\rho}^{\zeta} G(u)\log r^2(u\zeta')\,du$$

$$= \tfrac{1}{4}\int U(\zeta)\,G(\zeta)\,\delta\zeta\,d\psi.$$

Therefere $$\delta V = \tfrac{1}{2}\int U(\zeta)\,G(\zeta)\,\delta\zeta\,d\psi. \qquad (25.3)$$

But since the volume of the figure F is constant,

$$\int G(\zeta)\,\delta\zeta\,d\psi = 0;$$

also $$\int (x^2+y^2)\,G(\zeta)\,\delta\zeta\,d\psi = \int (\rho + q\cos^2\psi + \zeta)\,G(\zeta)\,\delta\zeta\,d\psi = 2\delta S.$$

Let us now use the condition that on the surface of F we have

$$U(\zeta) + (\Omega_0 + \eta)(x^2+y^2) = A\cos m\psi + \text{constant.}$$

Then we get

$$\delta V = \tfrac{1}{2}\int U(\zeta)\,G(\zeta)\,\delta\zeta\,d\psi$$

$$= \tfrac{1}{2}\int G(\zeta)\,\delta\zeta\,d\psi\,[\text{constant} + A\cos m\psi - (\Omega_0+\eta)(x^2+y^2)]$$

or

$$\delta V = \tfrac{1}{2}A\int G(\zeta)\cos m\psi\,\delta\zeta\,d\psi - (\Omega_0+\eta)\,\delta S. \qquad (25.4)$$

Now the left-hand side is an exact differential.

Therefore so must the right-hand sida be. But

$$\delta\zeta = \frac{\partial\zeta}{\partial a}\delta a + \frac{\partial\zeta}{\partial\eta}\partial\eta\,;\ \delta S = \frac{\partial S}{\partial a}\delta a + \frac{\partial S}{\partial\eta}\delta\eta.$$

Therefore the right-hand side

$$= \left[\tfrac{1}{2}A\int G(\zeta)\cos m\psi\,\frac{\partial\zeta}{\partial a}\,d\psi - (\Omega_0+\eta)\frac{\partial\zeta}{\partial a}\right]\delta a$$

$$+ \left[\tfrac{1}{2}A\int G(\zeta)\cos m\psi\,\frac{\partial\zeta}{\partial\eta}\,d\psi - (\Omega_0+\eta)\frac{\partial S}{\partial\eta}\right]\delta\eta.$$

If this is a perfect differential, we must have

$$\frac{\partial}{\partial\eta}\left[\text{co-efficient of }\delta a\right] = \frac{\partial}{\partial a}\left[\text{co-efficient of }\delta\eta\right].$$

Therefore we get the equation

$$\frac{\partial S}{\partial \alpha} = \tfrac{1}{2}\frac{\partial A}{\partial \eta}\int G(\zeta)\cos m\psi\,\frac{\partial \zeta}{\partial \alpha}\,d\psi$$

$$-\tfrac{1}{2}\frac{\partial A}{\partial \alpha}\int G(\zeta)\cos m\psi\,\frac{\partial \zeta}{\partial \eta}\,d\psi. \qquad (25.5)$$

This is the equation from which we shall deduce the values of S_{20}, S_{11}.

Consider the terms of the first degree in the expansion of the right-hand side of (25.5).

By (23.4) we know that

$$A \equiv A_2\,\alpha^2 + A_3\,\alpha^3 + \ldots\ldots + (B + \Sigma)\,\alpha\eta + C_3\,\eta^3 + \ldots\ldots$$

where $A_2, A_3 \ldots\ldots B, C_3, C_4 \ldots\ldots$

are definite constants and Σ is a power series in α, η vanishing for

$$\alpha = 0 = \eta.$$

Hence we need consider only the values of

$$\int \cos m\psi\,.\,G(\zeta)\,.\,\frac{\partial \zeta}{\partial \alpha}\,d\psi\;;\quad \int \cos m\psi\,.\,G(\zeta)\,.\,\frac{\partial \zeta}{\partial \eta}\,d\psi$$

for $\alpha = 0 = \eta$.

But from (21.5) we have $H\zeta_{10} = \cos m\psi$; and by (20.6) the ζ_{0s} are constants.

Since $m > 2$, we get

$$\int G_0 \cos m\psi\;\zeta_{10}\,d\psi = \int \frac{H(\rho\psi)}{\Delta(\rho)}\,\zeta_{10}\cos m\psi\;d\psi = \frac{\pi}{\Delta};$$

and $$\int G_0\,\zeta_{01}\cos m\psi\;d\psi = 0.$$

Therefore the terms of the first degree on the right-hand side of (25.5) are

$$\tfrac{1}{2}\Big[B\alpha\Big]\frac{\pi}{\Delta} - \tfrac{1}{2}\Big[2A_2\,\alpha + B\eta\Big]\times 0 = \tfrac{1}{2}\,\frac{\pi}{\Delta}\,B\alpha$$

On the left-hand side up to the terms of the first degree

$$\frac{\partial S}{\partial \alpha} = 2\,S_{20}\,\alpha + S_{11}\,\eta.$$

Comparing the two, we get the required expressions for S_{20}, S_{11}, *viz.*

$$S_{20} = \tfrac{1}{2}\,\frac{\pi}{\Delta}\,B;\quad S_{11} = 0.$$

Hence the expansion of S up to the terms of the second order is

$$S = S_0 + S'_0\,\eta + \tfrac{1}{4}\,\frac{\pi}{\Delta}\,B\,\alpha^2 + \tfrac{1}{2}\,S''_0\,\eta^2,$$

since S_{02} is obviously $= \frac{1}{2}\,\frac{d^2S}{d\Omega^2}$ for $\Omega = \Omega_0$.

Let us note that there is no term in α.

§ 26. **The Angular Momentum.**

The angular momentum is equal to $S\omega$. Therefore if

$$M = S^2\,\Omega \equiv \frac{\omega^2 S^2}{2}, \tag{26.1}$$

then $\qquad M \propto$ square of the angular momentum.

Let us find the expansion of M up to the second order terms in α, η. We have, for the figure F,

$$M = \Omega S^2 \equiv (\Omega_0 + \eta)\left[S_0 + S'_0\,\eta + \tfrac{1}{4}\,\frac{\pi}{\Delta}\,B\,\alpha^2 + \tfrac{1}{2}\,S''_0\,\eta^2 + \ldots\ldots\right]^2$$

$$= (\Omega_0 + \eta)\left[S_0^2 + 2S_0\,S'_0\,\eta + \frac{\pi}{2\Delta}\,B\,S_0\,\alpha^2 + S_0\,S''_0\,\eta^2 + S'^2_0\,\eta^2 + \ldots\ldots\right]$$

$$= \Omega_0\,S_0^2 + (S_0^2 + 2\,\Omega_0\,S_0\,S'_0)\,\eta + \frac{\pi}{2\Delta}\,\Omega_0\,S_0\,B\alpha^2 + (\ldots\ldots)\,\alpha^2\,\eta + (\ldots\ldots)\,\eta^2 + \ldots\ldots$$

$$= M_0 + M'_0\,\eta + \frac{\pi}{2\Delta}\,\Omega_0\,S_0\,B\alpha^2 + (\ldots\ldots)\,\alpha^2\,\eta + (\ldots\ldots)\eta^2 + \ldots\ldots \tag{26.2}$$

Let us note that here also there is no term in α.

§ 27. Now, when the figure F is one of equilibrium, we must have the parameters α, η connected by the equation

$$0 = A \equiv A_2\,\alpha^2 + A_3\,\alpha^3 + \ldots\ldots + (B + \Sigma)\,\alpha\eta + C_3\,\eta^3 + \ldots\ldots$$

and the definitive expression for ζ is got by solving this equation for one or other of the parameters and substituting that value in

$$\zeta = \sum_{r+s>0} \zeta_{rs}\,\alpha^r\,\eta^s$$

Since, as we shall soon show, B is never zero this can be done in two ways:—

(1) we may be able to solve for α in terms of η,

or (2) we may be able to solve for η in terms of α.

We shall now show that (1) gives only elliptical figures of equilibrium near the given cylinder, whereas (2) gives non-elliptical figures of equilibrium of the type F.

To prove this, we need only remark that, if α is a function of η, then we get for the definitive expression for ζ

$$\zeta = a_1'\eta + a_2\eta^2 + \ldots\ldots$$

We shall compare this value with the value given for elliptical figures of equilibrium near the given cylinder.

From (20.3, 20.4) we see that for them

$$\zeta = \frac{d\rho}{d\Omega}\eta + \ldots\ldots\ldots + \frac{1}{s!}\eta^s\frac{d^s\rho}{d\Omega^s} + \ldots\ldots\ldots,$$

whereas for the non-elliptical figures (near the given cylinder) which correspond to the same angular velocity as the given cylinder, the series for ζ is of the form

$$\zeta = \zeta_1\alpha + \zeta_2\alpha^2 + \ldots\ldots\ldots\ldots$$

Hence, when we are considering the non-elliptical figures of equilibrium the equation $A = 0$ is to be solved for η in terms of α.

Now, from what we have shown already, we see that the constant B can never be zero; for we have shown by comparing the two methods of defining the adjacent figure (at that stage not necessarily one of equilibrium) that when we make this last a figure of equilibrium, the conditions for the same are identical at the point of bifurcation given by $T_m = 0$, *viz.*

$$L = 0 \text{ and } A = 0.$$

In fact, when $T_m = 0$, we have from (21.2)

$$L\alpha \equiv \frac{\Delta}{\pi^2}\int W\cos m\psi\, d\psi = \frac{\Delta}{\pi}A.$$

Hence comparing the co-efficients of the various terms $\alpha^r\eta^s$, we get

$$L_{01} = \text{co-efficient of } \alpha\eta \text{ in } L\alpha = \frac{\Delta}{\pi}B.$$

But from (20.8) $$L_{01} = -\frac{dT_m}{d\Omega}.$$

Hence $$B = -\frac{\pi}{\Delta}\frac{dT_m}{d\Omega}.$$

But from (10.5), we have

$$T_n = \tfrac{1}{2}(1-t^2) - \frac{1}{n}(1+t^n)\,;\ \Omega = \frac{\omega^2}{2} = \frac{2\pi ab}{(a+b)^2} = \frac{\pi}{2}(1-t^2).$$

Therefore $$\frac{dT_m}{d\Omega} = \frac{dT_m}{dt}\cdot\frac{dt}{d\Omega} = \frac{1+t^{m-2}}{\pi}, \qquad (27.2)$$

from which it is obvious that $\frac{dT_m}{d\Omega}$ can never be zero but is always positive*. Hence the equation $T_m = 0$ has only a single root to which corresponds the elliptic cylinder at the point of bifurcation.

Further, we see that B is always different from zero being in fact negative.

Thus for non-elliptical figures of equilibrium, we get the expansion for η in the form

$$\left.\begin{aligned} \eta &= -\frac{A_2}{B}\alpha + (\ldots\ldots)\alpha^2 + \ldots\ldots \text{ if } m \text{ is even,} \\ \text{and } \eta &= -\frac{A_3}{B}\alpha^2 + (\ldots\ldots)\alpha^3 + \ldots\ldots \text{ if } m \text{ is odd;} \end{aligned}\right\} \qquad (27.3)$$

for, as we shall show later, $A_2 \equiv 0$ when m is odd.

Therefore if m is even

$$\left.\begin{aligned} M &= M_0 - M'_0\frac{A_2}{B}\alpha + (\ldots\ldots)\alpha^2 + \ldots\ldots \\ &\text{and if } m \text{ is odd} \\ M &= M_0 + \left(\frac{\pi}{2\Delta}\Omega_0 S_0 B - M'_0\frac{A_3}{B}\right)\alpha^2 + \ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (27.4)$$

---

* For, the possible root $t = -1$ must be rejected since t always lies between 0 and 1 (including the extreme values).

§ 28. Now for the elliptic cylinders, we know that

$$S = \frac{\pi ab(a^2 + b^2)}{4}; \quad \omega^2 = \frac{4\pi ab}{(a + b)^2} \text{ or } \Omega = \frac{2\pi ab}{(a + b)^2}.$$

$$\therefore \quad M \equiv \frac{S^2\omega^2}{2} = \frac{\pi^3 a^3 b^3}{8} \frac{(a^2 + b^2)^2}{(a + b)^2};$$

we want

$$S'_0 \text{ and } M'_0; \text{ or } \left(\frac{dS}{d\Omega}\right)_{\Omega = \Omega_0}, \left(\frac{dM}{d\Omega}\right)_{\Omega = \Omega_0}.$$

Since the volume is constant,

$$ab = \text{constant} = c^2 \text{ say.}$$

Thus

$$\frac{dS}{da} = \frac{d}{da}\left[\frac{\pi c^2}{4}\left(a^2 + \frac{c^4}{a^2}\right)\right] = \frac{\pi b}{2a}(a^2 - b^2)$$

and is therefore positive; that is, the moment of inertia increases as a increases.

Similarly

$$\frac{dM}{da} = \frac{d}{da}\left\{\frac{\pi^3 c^3}{8} \frac{\left(a^2 + \frac{c^4}{a^2}\right)^2}{\left(a^2 + \frac{c^2}{a}\right)^2}\right\}$$

$$= \frac{\pi^3 c^3 (a^4 + c^4)}{4a^3 (a^2 + c^2)^3}(a^2 - c^2)(a^4 + 4a^2c^2 + c^4)$$

and is therefore positive; that is, the angular momentum increases as a increases.

But $$\frac{d\Omega}{da} = 2\pi \frac{d}{da}\left\{\frac{c^2}{\left(a + \frac{c^2}{a}\right)^2}\right\}$$

$$= \frac{4\pi a c^2}{(a^2 + c^2)^3}(c^2 - a^2)$$

and is therefore negative, that is, the angular velocity *decreases* as a increases.

Thus S'_0 and M'_0 are both negative for the elliptic cylinders. (28.1)

§ 29. Let us now go back to equations (27.4) which give us the angular momentum of the figure F.

We have shown that

$$B = -\frac{\pi}{\Delta}\frac{dT_m}{d\Omega}. \qquad (27.1)$$

Hence using (27.2), we get

$$B = -\frac{1 + t^{n-2}}{\Delta} = -\frac{1 + t^{n-2}}{ab} \tag{29.1}$$

and is negative.* If now we consider the case of m even, the first of the equations (27.4) shows that $(M - M_0)$ cannot always be of the same sign; for it changes sign with α.

Therefore in the case of m even, the angular momentum of F is sometimes greater and sometimes less than that of the given cylinder. This would mean that all the cylinders given by the points of bifurcation for which m is even, are unstable.

Consider now the case of m odd:

Since B is negative and M'_0 negative, we see that $(M - M_0)$ is certainly negative if A_3 be positive.

If A_3 be negative, we cannot say *a priori* what the sign of $(M - M_0)$ will be.

Thus for *instability* it is *sufficient* that A_3 be positive.

We shall show later on that A_3 is positive for $m = 3$. It is probably so for all values of m.

§ 30. In the above considerations as regards the stability of the figure F we have assumed that there will be a minimum of energy for that figure if its angular momentum is greater than the angular momentum of the cylinder from which it is derived. We now proceed to prove that this is true.†

We know that the pear-shaped figures form a continuous series of figures of equilibrium (non-elliptical) and are defined by the parameter α. For the elliptic cylinder series $\alpha = 0$. Let F_0 be a pear-shaped figure of equilibrium defined by $\alpha = \alpha_0$ where $|\alpha_0|$ is small, since the pear-shaped figure is derived from an elliptic cylinder for which $\alpha = 0$. We wish to find out under what circumstances there is a minimum of energy for the figure F_0.

In our notation, this energy is

$$\frac{M_{\alpha_0}}{S} - V.$$

* We shall later on verify that B has actually this value which we have got from more or less *a priori* considerations.

† See note 2 on p. 6 of the Introduction,

Let us consider the variation of the energy due to variations in α, η on which the energy depends.

Since we are considering the case of equilibrium when the angular momentum is kept constant, we shall now find the necessary and sufficient conditions for such equilibrium.

$$0 = \delta_1(\text{Energy}) = -\frac{M_{\alpha_0}}{S^2}\delta S - \delta V.$$

But from equation (25.4), we have

$$\delta V = \tfrac{1}{2} A \int G(\zeta_0) \cos m\psi \, \delta\zeta \, d\psi - (\Omega_0 + \eta)\,\delta S$$
$$= A\delta\phi - (\Omega_0 + \eta)\,\delta S$$

if $$\phi = \tfrac{1}{2}\int_0^{2\pi} \cos m\psi \, d\psi \int_0^{\zeta} G(u)\,du. \tag{30.1}$$

$$\therefore \quad \delta_1(\text{Energy}) = \left[(\Omega_0 + \eta) - \frac{M_{\alpha_0}}{S^2}\right]\delta S - A\delta\phi \tag{30.2}$$

$$= \frac{M - M_{\alpha_0}}{S^2}\delta S - A\delta\phi.$$

This must reduce to zero whatever $\delta\alpha$, $\delta\eta$ be in order that there may be equilibrium and this is true for the figure F_0 for which

$$M = M_{\alpha_0}; \quad A = 0. \tag{30.3}$$

To determine the stability, we want the second increment of the energy. But using (30.2) and noting that for equilibrium of F_0 we have (30.3), *i.e.*,

$$M = M_{\alpha_0}; \quad A = 0 \text{ or } (\Omega_0 + \eta)\,S^2 = M_{\alpha_0}; \; A = 0,$$

we get

$$\delta_2(\text{Energy}) = \frac{\delta M \,\delta S}{S^2} - \delta A\,\delta\phi. \tag{30.4}$$

Thus the right-hand side of this equation will reduce to a quadratic form in $\delta\alpha$, $\delta\eta$ and this form must always be positive for stability.

Let us now introduce instead of $\delta\eta$ another arbitrary quantity ε defined by

$$\delta\eta = \eta'\delta\alpha + \varepsilon, \tag{30.5}$$

where $$\eta' = \frac{d\eta_\alpha}{d\alpha},$$

η being equal to η_α which satisfies the equation

$$A = 0.$$

From the equation $A = 0$, we get

$$\frac{\partial A}{\partial \alpha} + \frac{\partial A}{\partial \eta}\eta' = 0; \tag{30.6}$$

we have also

$$\frac{\partial M}{\partial \alpha} + \frac{\partial M}{\delta \eta}\eta' = M'; \frac{\partial S}{\partial \alpha} + \frac{\partial S}{\partial \eta}\eta' = S'; \frac{\partial \phi}{\phi \alpha} + \frac{\partial \phi}{\partial \eta}\eta' = \phi'.$$

$$\therefore \quad \delta M = \frac{\partial M}{\partial \alpha}\partial \alpha + \frac{\partial M}{\partial \eta}\partial \eta = \frac{\partial M}{\partial \eta}\varepsilon + M'\delta \alpha.$$

Similarly

$$\delta S = \frac{\partial S}{\partial \eta}\varepsilon + S'\delta \alpha$$

$$\delta \phi = \frac{\partial \phi}{\partial \eta}\varepsilon + \phi'\delta \alpha$$

$$\delta A = \frac{\partial A}{\partial \eta}\varepsilon.$$

Hence we get from (30.4)

$$\delta_2 \,(\text{Energy}) = \frac{1}{S^3}\left[P\varepsilon^2 + Q\varepsilon\,\delta\alpha + M'S'(\delta\alpha)^2\right] \tag{30.7}$$

where

$$\left.\begin{aligned} P &= \frac{\partial M}{\partial \eta}\cdot\frac{\partial S}{\partial \eta} - S^2\frac{\partial A}{\partial \eta}\cdot\frac{\partial \phi}{\partial \eta}. \\ Q &= S'\frac{\partial M}{\partial \eta} + M'\frac{\partial S}{\partial \eta} - S^2\phi'\frac{\partial A}{\partial \eta}. \end{aligned}\right\} \tag{30.8}$$

The expression for Q can be simplified a little.

From (25.5) and (30.1), we have

$$\frac{\partial S}{\partial \alpha} = \frac{\partial A}{\partial \eta}\cdot\frac{\partial \phi}{\partial \alpha} - \frac{\partial A}{\partial \alpha}\cdot\frac{\partial \phi}{\partial \eta}.$$

Therefore from (30.6) it follows that

$$\frac{\partial S}{\partial \alpha} = \phi'\frac{\partial A}{\partial \eta} = S' - \eta'\frac{\partial S}{\partial \eta}. \tag{30.9}$$

$$\therefore \quad S'\frac{\partial M}{\partial \eta} - S^2\phi'\frac{\partial A}{\partial \eta} = \left(\frac{\partial M}{\partial \eta} - S^2\right)S' + S^2\eta'\frac{\partial S}{\partial \eta}.$$

But

$$M = (\Omega + \eta)\,S^2.$$

$$\therefore \quad \frac{\partial M}{\partial \eta} = S^2 + 2(\Omega_0 + \eta) S \frac{\partial S}{\partial \eta}; \; M' = S^2\eta' + 2(\Omega_0 + \eta) SS',$$

and
$$\left(\frac{\partial M}{\partial \eta} - S^2\right) S' = \left(M' - S^2\eta'\right) \frac{\partial S}{\partial \eta}. \qquad (30.10)$$

$$\therefore \quad Q = 2M' \frac{\partial S}{\partial \eta}.$$

Thus to make the energy a minimum, we have to make

$$P\varepsilon^2 + 2M' \frac{\partial S}{\partial \eta} \varepsilon \delta\alpha + M'S' (\delta\alpha)^2$$

a positive form, P being given by (10.3)

Now we have from (25.6), (26.2)

$$S = S_0 + S'_0\eta + \tfrac{1}{4} \frac{\pi}{\Delta} B\alpha^2 + \tfrac{1}{2} S''_0\eta^2 + \ldots\ldots$$

$$M = M_0 + M'_0\eta + \frac{\pi}{2\Delta} \Omega_0 S_0 B\alpha^2$$
$$+ [\Omega_0 (S_0S''_0 + S'^2_0) + 2S_0S'_0]\eta^2.$$

Therefore, to the first order in α, η we have

$$\frac{\partial S}{\partial \alpha} = \frac{\pi}{2\Delta} B\alpha; \; \frac{\partial S}{\partial \eta} = S'_0$$

$$\frac{\partial M}{\partial \alpha} = \frac{\pi}{\Delta} \Omega_0 S_0 B\alpha; \; \frac{\partial M}{\partial \eta} = M'_0.$$

Hence for $\alpha = 0 = \eta$,

$$\left.\begin{aligned} \frac{\partial S}{\partial \alpha} = 0; \; \frac{\partial S}{\partial \eta} = S'_0, \\ \frac{\partial M}{\partial \alpha} = 0; \; \frac{\partial M}{\partial \eta} = M'_0. \end{aligned}\right\} \qquad (30.11)$$

$\therefore$ $M' = M'_0\eta'$; $S' = S'_0\eta'$ for $\alpha = 0 = \eta$.

$\therefore$ $M'S' = M'_0S'_0\eta'^2$ + terms of higher order in α, η.

But we have already shown (28.1) that M'_0 and S'_0 are both negative.

Therefore $M'S'$ is positive. (30.12)

But
$$P = \frac{\partial M}{\partial \eta} \cdot \frac{\partial S}{\partial \eta} - S^3 \frac{\partial A}{\partial \eta} \cdot \frac{\partial \phi}{\partial \eta}.$$

Therefore from (30.11), as $\frac{\partial A}{\partial \eta} = 0$ for $\alpha = 0 = \eta$, we see that

$$P = M'_0 S'_0$$

when $\alpha = 0 = \eta$ and is therefore positive.

Thus the co-efficients of the square terms in the quadratic form are positive.

Hence if we have

$$M' \left[PS' - M' \left(\frac{\partial S}{\partial \eta} \right)^2 \right] > 0, \qquad (30.13)$$

the quadratic form will always be positive and there will be stability.

But

$$PS' - M' \left(\frac{\partial S}{\partial \eta} \right)^2 \equiv \left(S' \frac{\partial M}{\partial \eta} - M' \frac{\partial S}{\partial \eta} \right) \frac{\partial S}{\partial \eta} - S^2 S' \frac{\partial A}{\partial \eta} \frac{\partial \phi}{\partial \eta};$$

and from (30.10)

$$S' \frac{\partial M}{\partial \eta} - M' \frac{\partial S}{\partial \eta} = S^2 \left(S' - \eta' \frac{\partial S}{\partial \eta} \right) = S^2 \frac{\partial S}{\partial \alpha}$$

$$= S^2 \phi' \frac{\partial A}{\partial \eta} \text{ from (30.9)};$$

also $$\phi' \frac{\partial S}{\partial \eta} - S' \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial S}{\partial \eta} - \frac{\partial S}{\partial \alpha} \cdot \frac{\partial \phi}{\partial \eta}. \qquad (30.6)$$

Hence the inequality (30.13) becomes

$$M' \frac{\partial A}{\partial \eta} \left[\frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial S}{\partial \eta} - \frac{\partial S}{\partial \alpha} \cdot \frac{\partial \phi}{\partial \eta} \right] > 0. \qquad (30.14)$$

Consider now the expansion of the left-hand side of (30.14) as far as the first order terms are concerned; for unless they are identically equal to zero the sign of the expression will depend on the first order terms.

Now $$\frac{\partial A}{\partial \eta} = B\alpha + \ldots\ldots$$

and since $$\phi = \tfrac{1}{2} \int \cos m\psi \, d\psi \int_0^{\zeta} G(u)\, du,$$

$$\frac{\partial \phi}{\partial \alpha} = \tfrac{1}{2} \int G(\zeta) \cos m\psi \frac{\partial \zeta}{\partial \alpha} \, d\psi.$$

The absolute term in this is equal to

$$\tfrac{1}{2} \int G_0 \zeta_{10} \cos m\psi \, d\psi = \tfrac{1}{2} \cdot \frac{1}{\Delta} \int H \zeta_{10} \cos m\psi \, d\psi$$

$$= \frac{\pi}{2\Delta}.$$

$$\therefore \quad \frac{\partial \phi}{\partial \alpha} = \frac{\pi}{2\Delta} + \ldots\ldots\ldots$$

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NOTES AND QUESTIONS

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Notes and Questions.

Cyclotomic Sexi-Section.*

If we remove the obvious root 1 from the binomial equation

$$x_n - 1 = 0$$

by dividing by $x - 1$, we obtain

$$x^{n-1} + x^{n-2} + \dots\dots + x^3 + x + 1 = 0. \qquad \dots \quad (1)$$

If n be prime, the irreducible equation (1) is called a *cyclotomic equation.* Karl Friedrich Gauss has developed a remarkable scheme in his *Disquisitiones Arithmeticæ* by which the solution of this equation is made to depend on other equations called 'Equations of Periods' whose degrees are the prime-factors of $n - 1$.

Let $n - 1 = ef$, and ω any primitive root of (1). Then the roots of (1) can be expressed as

$$\omega,\ \omega^g,\ \omega^{g^2},\ \omega^{g^3},\ \dots\dots\ \omega^{g^r},\ \dots\dots\ \omega^{g^{n-2}}$$

where g is an integer (called a *primitive root of n*) such that

$$g,\ g^2,\ g^3,\ \dots\dots\ g^r,\ \dots\dots\ g^{n-2},\ g^{n-1}$$

when divided by n leave different remainders.

Following Gauss's theory, we may divide the roots of (1) into periods as follows:

$$\eta_0 = \omega \quad + \omega^{g^e} \quad + \omega^{g^{2e}} \quad + \dots\dots + \omega^{g^{(f-1)e}}$$

$$\eta_1 = \omega^g \quad + \omega^{g^{e+1}} + \omega^{g^{2e+1}} + \dots\dots + \omega^{g^{(f-1)e+1}}$$

....

$$\eta_{e-1} = \omega^{g^{e-1}} + \omega^{g^{2e-1}} + \omega^{g^{3e-1}} + \dots\dots + \omega^{g^{fe-1}}$$

These quantities η are called the f-nomial periods and it is shown by Gauss that these periods are the roots of an irreducible integral equation of the eth degree which can be solved by radicals. This equation of the eth degree is called the *equation of periods.*

The quadratic and the cubic equation of periods have been discussed at length in Gauss's memoir itself, while the quartic equation has been

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* Read at the Eleventh Session of the Indian Science Congress, Bangalore, 1924.

considered by Arthur Cayley in the *Proceedings of the London Mathematical Society*, 1st Series, Vol. II, (1880). Recently, however, in 2nd Series, Vol. 14, (1915) of the same *Proceedings*, Prof W. Burnside completely resolved the problem of quinqui-section, which had been solved previously by Prof. Rogers, whose work, W. Burnside regrets, had entirely slipped from his memory.

In this paper I make an attempt to solve the corresponding general problem for sexi-section, believing that it has not been dealt with before. In passing, I may note that Pandit Oudh Upadhyaya, Research Scholar, Lucknow University, has computed the sextic equation of periods for the eight primes of the form $6p+1$ from 13 to 73 by laboriously computing the symmetric functions of the roots by actual multiplication. (Vide: *The Tohoku Mathematical Journal*, Vols. 21 and 22).

The method followed here is in some respects analogous to that adopted by G. B. Mathews for deriving the cubic equation of periods in his *Theory of Numbers*, Part I, pp. 219—223.

When $n = 6p + 1$, there will be a set of six periods

$$\eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5$$

which are the roots of a sextic equation:

$$x^6 + c_1x^5 + c_2x^4 + c_3x^3 + c_4x^2 + c_5x + c_6 = 0$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are integers to be determined.

Let g be any primitive root of n and $\omega = e^{\frac{2\pi i}{n}}$ as usual; put

$$\eta_0 = \Sigma\, \omega^{g^{6h}}, \quad \eta_1 = \Sigma\, \omega^{g^{6h+1}}, \quad \eta_2 = \Sigma\, \omega^{g^{6h+2}}$$

$$\eta_3 = \Sigma\, \omega^{g^{6h+3}}, \quad \eta_4 = \Sigma\, \omega^{g^{6h+4}}, \quad \eta_5 = \Sigma\, \omega^{g^{6h+5}}$$

All numbers less than and prime to n may be distributed into 6 classes

$$\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma'$$

according as they are indices of the powers of ω in

$$\eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4, \text{ or } \eta_5.$$

We observe that 1 and $n - 1 \equiv -1$ both belong to the class 'α' when n is of the form $12m + 1$, but $n - 1 \equiv -1$ belongs to the class 'α'' when n is of the form $12m + 7$. We give below the

multiplication-table giving the classes to which the products of pairs of numbers less than n are severally congruent:

Cl.	α	α'	β	β'	γ	γ'
α	α	α'	β	β'	γ	γ'
α'	α'	α	β'	β	γ'	γ
β	β	β'	γ	γ'	α'	α
β'	β'	β	γ'	γ	α	α'
γ	γ	γ'	α'	α	β'	β
γ'	γ'	γ	α	α'	β	β'

We can easily infer from the above multiplication-table the congruences equivalent to the congruences of the types

$$\text{(i)}\quad 1 + M + N \equiv 0 \pmod{n}$$
$$\text{(ii)}\quad M + N \equiv 0 \pmod{n}$$

where M and N denote numbers belonging to one or other of the six classes mentioned above.

Denoting by capital A's, B's, C's with numeral subscripts, the number of distinct solutions of congruences of type (i) and by capital D's with numeral subscripts the number of distinct solutions of congruences of type (ii), we may exhibit the scheme of symbols to denote equivalent congruences thus:

M (columns), N (rows)

N \ M	α	α'	β	β'	γ	γ'
α	A_2	A	B_1	C_2	C_3	B_2
α'	A	A_1	B	B	C	C
β	B_1	B	B_2	C	C_1	C_1
β'	C_2	B	C	C_3	B_3	C_1
γ	C_3	C	C_1	B_3	C_2	B
γ'	B_2	C	C_1	C_1	B	B_1

M (columns), N (rows)

N \ M	α	α'	β	β'	γ	γ'
α	D_1	D	D_2	D_3	D_3	D_2
α'	D	D_1	D_3	D_3	D_2	D_3
β	D_2	D_3	D_1	D	D_3	D_3
β'	D_3	D_3	D	D_1	D_3	D_2
γ	D_3	D_2	D_3	D_3	D_1	D
γ'	D_2	D_3	D_3	D_2	D	D_1

In the above table, the capital letters M and N stand for the numbers belonging to the classes in the top-row and the left column respectively, to be selected to form the congruences of types (i) and (ii).

Thus A, B, C, D denote respectively the number of distinct solutions of the congruences:

$$1 + \alpha + \alpha' \equiv 0,\ 1 + \gamma + \gamma' \equiv 0,\ 1 + \beta + \beta' = 0,\ \alpha + \alpha' \equiv 0,$$

and their equivalents.

We may now write the following fundamental identities:—

$$\eta_0 \eta_3 = D + A(\eta_0 + \eta_3) + B(\eta_1 + \eta_4) + C(\eta_2 + \eta_5)$$

$$\eta_1 \eta_4 = D + C(\eta_0 + \eta_3) + A(\eta_1 + \eta_4) + B(\eta_2 + \eta_5)$$

$$\eta_2 \eta_5 = D + B(\eta_0 + \eta_3) + C(\eta_1 + \eta_4) + A(\eta_2 + \eta_5)$$

$$\therefore\ \Sigma\, \eta_0 \eta_3 = 3D + \Sigma A . \Sigma\, \eta_0 = 3D - \Sigma A.$$

For convenience, let us denote

$$\eta_0 + \eta_3,\quad \eta_1 + \eta_4,\quad \eta_2 + \eta_5.$$

by X, Y, Z respectively and

$$A - D,\quad B - D,\quad C - D$$

by A′, B′, C′ respectively. Then we may re-write the above relations in the form

$$\left.\begin{aligned} \eta_0 \eta_3 &= A'X + B'Y + C'Z \\ \eta_1 \eta_4 &= C'X + A'Y + B'Z \\ \eta_2 \eta_5 &= B'X + C'Y + A'Z \\ \Sigma\, \eta_0 \eta_3 &= -\Sigma A' \end{aligned}\right\} \quad \ldots \quad \ldots \quad (I)$$

remembering that

$$X + Y + Z = \Sigma\, \eta_0 = -1.$$

Now, we proceed to calculate ce-efficients 'c.' Obviously from equation (1)

$$c_1 = -\Sigma\, \eta_0 = 1 \qquad \ldots \quad (II)$$

$$c_2 = \Sigma\, XY + \Sigma\, \eta_0 \eta_3.$$

But X, Y, Z are the roots of the cubic equation of periods

$$y^3 + y^2 - \tfrac{1}{3}(n-1)y - \tfrac{1}{9}\left(na' + \frac{n-1}{3}\right) = 0,$$

where a' is the integer obtained by equating $3a' - 2$ to a suitable value of $\sqrt{a^2}$, when $4n$ is represented in the form $a^2 + 27b^2$.

$$\therefore\ \Sigma\, XY = -\tfrac{1}{3}(n-1) = \text{an integer.}$$

Since c_2 should also be an integer, $\Sigma\,\eta_0\eta_3$ should be integral, so that we may write

$$3D - \Sigma A = \Sigma\,\eta_0\,\eta_3 = 3D - \frac{3p^2 - 3D}{6p}$$

$$\therefore\quad \Sigma A = \frac{n-1}{12} - \frac{3D}{n-1}.$$

But $D = 0$ when n is of the form $12m + 1$; for, since gNn, the classes α, β', γ contain purely quadratic residues while α', β, γ' contain only non-residues, so that

$$\alpha + \alpha' \not\equiv 0 \pmod{n}.$$

When n is of the form $12\,m + 7$, for every number α_r in α there is a corresponding number α'_r in α' such that $\alpha_r + \alpha'_r \equiv 0$; hence

$$D = 2m + 1 = \frac{n-1}{6}.$$

$$\therefore\quad \Sigma A' = \frac{n-1}{12},\ \text{ or } -\frac{5n+1}{12},$$

according as n is of the form $12\,m + 1$, or $12\,m + 7$.

$$\therefore\quad c_2 = -\frac{5(n-1)}{12},\ \text{ or } \frac{n+5}{12}$$

according as n is of the form $12\,m + 1$, or $12\,m + 7$. ... (III)

Next, $-c_3 = XYZ - \Sigma\,\eta_0\,\eta_3 - (X\eta_0\,\eta_3 + Y\eta_1\,\eta_4 + Z\eta_2\,\eta_5)$

$$= \frac{1}{9}\left(na' + \frac{n-1}{3}\right) + \frac{n+2}{3}\,.\,\Sigma A' - nA'$$

$$= \frac{1}{9}\,na' + \frac{(n-1)(3n+10)}{108} - nA$$

when n is of the form $12\,m + 1$.

When n is of the form $12\,m + 7$, the result is

$$\frac{1}{9}\,na' + \frac{3n^2 - 47n - 10}{108} - nA. \qquad \text{.. (IV)}$$

The value of the next co-efficient

$c_4 = \Sigma\,\eta_0\eta_3\,YZ + \Sigma\,(A'X + B'Y + C'Z)\,(B'X + C'Y + A'Z)$

$$= \tfrac{1}{3}\,(na' + \frac{n-1}{3}\,A') + B'\xi + C'\zeta + n\,\Sigma\,A'B'$$

$$- \tfrac{1}{3}\,(n-1)\,(\Sigma\,A')^2$$

where ξ, ζ stand for the cyclic functions

$$Y^2Z + Z^2X + X^2Y \text{ and } YZ^2 + ZX^2 + XY^2$$

respectively.

Now $\quad \xi + \zeta = \Sigma\,X\,(Y^2 + Z^2) = \frac{1}{3}\left[\frac{2(n-1)}{3} - na'\right]$

and $$\xi - \zeta = \sqrt{D}$$
$$= \pm nb,$$

where D is the discriminant of the cubic equation of periods.

We may, for the present, include the proper sign in b and write

$$\xi - \zeta = nb,$$

where the sign of b will depend on the choice of the primitive root g and therefore cannot be determined *a priori.*

Substituting the values of ξ and ζ in c_4 and reducing the expression, we get

$$c_4 = n\,\Sigma\,AB + \tfrac{1}{2}\,nb\,(B - C) + \tfrac{1}{2}\,n\,Aa' - \frac{n(n-1)}{72}\,a'$$
$$- \frac{(n-1)(n-5)}{432}, \text{ when } n \equiv 12\,m + 1;$$
$$= n\,\Sigma\,AB + \tfrac{1}{2}\,nb\,(B - C) + \tfrac{1}{2}\,n\,Aa' - \frac{n(n-7)}{72}\,a'$$
$$- \frac{(n-1)(n^2 - 18\,n + 5)}{432}, \text{ when } n \equiv 12\,m + 7. \quad \ldots \text{(V)}$$

Again,

$$-c_5 = \Sigma\,X\,(B'X + C'Y + A'Z)\,(C'X + A'Y + B'Z)$$
$$= -nB'C' + \tfrac{1}{3}\,(A'^2 + 2\,B'C')\left(na' + \frac{n-1}{3}\right)$$
$$+ (B'^2 + 2\,A'C')\,\zeta + (C'^2 + 2\,A'B')\,\xi$$
$$= n\,BC\,(a' - 1) + \tfrac{1}{2}\,na'\left(A^2 - \frac{(n-1)^2}{432}\right) + \tfrac{1}{2}\,nb\,(B - C) \times$$
$$\left(3A - \frac{n-1}{12} + \frac{(n-1)^3}{1296}\right), \text{ when } n \equiv 12\,m + 1;$$
$$= n\,BC\,(a' - 1) + \tfrac{1}{2}na'\left(A^2 - \frac{(n-7)^2}{432}\right)$$
$$+ \tfrac{1}{2}\,nb(B - C)\left(3A - \frac{n-7}{12}\right) - \frac{n(n-1)}{6}\,A$$
$$+ \frac{(n-1)(7n^2 - 80\,n + 1)}{1296}, \text{ when } n \equiv 12\,m + 7 \ldots \text{(VI)}$$

Lastly

$$c_6 = (A'X + B'Y + C'Z)\,(B'X + C'Y + A'Z)\,(C'X + B'Z + A'Y)$$
$$= -n\,A'B'C' + \frac{1}{9}\left(na' + \frac{n-1}{3}\right)(\Sigma\,A'^3 + 6A'B'C')$$
$$+ \xi\,(A'C'^2 + B'A'^2 + C'B'^2) + \zeta\,(A'C' + B'^2A' + C'^2B')$$
$$= \frac{(n-1)^4}{36^3} - \tfrac{1}{2}\,nb\;\Pi\;(A - B) + \tfrac{1}{2}\,n\,ABC\,(3a' - 2)$$
$$+ \frac{n(n-1)\,a'}{24}\left[\frac{(n-1)^2}{648} - \Sigma\,AB\right], \text{ when } n \equiv 12\,m + 1;$$

$$= \frac{(n-7)^3}{36^3}(3na'+n-1) - \frac{(n-1)^3(n-19)}{36^2 . 2} - \tfrac{1}{2} nb \, \Pi \, (A-B)$$

$$+ \frac{n}{C}(\Sigma \, AB)\left[(n-1) - \frac{n-7}{4}.a'\right] + \tfrac{1}{2} n ABC (3a'-2),$$

when $n \equiv 12m + 7$ ... ... (VII)

Finally, therefore, the sextic equation of periods is

$$x^6 + x^5 - \frac{5(n-1)}{12}x^4 - \left[\frac{1}{9}na' + \frac{(n-1)(3n+10)}{108} - An\right]x^3$$

$$+ \left[n \, \Sigma \, AB + \tfrac{1}{2} nb (B-C) + \tfrac{1}{2} n Aa' - \frac{n(n-1)}{72}a' - \frac{(n-1)^2(n-5)}{432}\right]x^2$$

$$- \left[n BC (a'-1) + \tfrac{1}{2} na' \left(A^2 - \frac{\overline{n-1}^2}{432}\right) + \tfrac{1}{2} nb (B-C)\left(3A - \frac{n-1}{12}\right) + \frac{(n-1)^3}{1296}\right]x$$

$$+ \left[\frac{(n-1)^4}{36^3} - \tfrac{1}{2} nb \, \Pi \, (A-B) + \tfrac{1}{2} n ABC (3a'-2) + \frac{n(n-1)a'}{24}\left[\frac{(n-1)^2}{648} - \Sigma \, AB\right]\right] = 0, \quad \text{... (VIII)}$$

or

$$x^6 + x^5 + \frac{n+5}{12}x^4 - \left[\frac{1}{9}na' + \frac{3n^2-47n-10}{108} - An\right]x^3$$

$$+ \left[n \, \Sigma \, AB + \tfrac{1}{2} nb (B-C) + \tfrac{1}{2} n Aa' - \frac{n(n-7)}{72}a' - \frac{(n-1)(n^2-18n+5)}{432}\right]x^2$$

$$- \left[n BC (a'-1) + \tfrac{1}{2} na' \left(A^2 - \frac{\overline{n-7}^2}{432}\right) + \tfrac{1}{2} nb (B-C)\left(3A - \frac{n-7}{12}\right) - \frac{n(n-1)}{6}A + \frac{(n-1)(7n^2-80n+1)}{1296}\right]x$$

$$+ \frac{(n-7)^3}{36^3}(3na'+n-1) - \frac{(n-1)^3(n-19)}{36^2 . 2} - \tfrac{1}{2} nb \, \Pi \, (A-B)$$

$$+ \frac{n}{6}\left(n - 1 - \frac{n-7}{4}a'\right)\Sigma \, AB + \tfrac{1}{2} n ABC (3a'-2) = 0,$$

according as n is a prime of the form $12m+1$, or $12m+7$. ... (IX)

Thus, the problem of cyclotomic sexi-section may be considered to be completely solved if we can find the sign of b as well as the constants A and B directly. But it is almost an impossible task to derive a Diophantine equation involving A and B only by eliminating the other 13 variables A_1, B_1 etc, from the several identities that we can get by considering the number of distinct solutions of congruences like

$$1 + \alpha + \beta + \gamma \equiv 0.$$

For particular primes, however, the values of A and B can be obtained by inspection, once we write down the numbers in the classes α', β, β'.

Illustration: Calculation of the sextic equation of periods for the prime 79.

Taking 3 as the primitive root, we get the numbers, belonging to α', β, β', respectively

(i) 12, 14, 15, 17, 27, 33, 41, 57, 58, 61, 69, 71, 78

(ii) 3, 24, 28, 30, 34, 35, 37, 43, 54, 59, 63, 66, 77

(iii) 2, 13, 16, 20, 25, 36, 42, 44, 45, 49, 51, 55, 76.

Now A = the number of pairs of consecutive numbers in (i)
= 2;

B = the number of numbers in the class (i) which, when increased by unity, fall into class (ii)
= 3.

Since $A + B + C = \frac{n-7}{12} = 6$, $C = 1$.

Expressing $4 \times 79 = 316$ in the form $a^2 + 27b^2$, we get

$$316 = 17^2 + 27.1^2,$$

whence we get $a' = -5$, $b = \pm 1$.

To clear the ambiguity in the sign of b, we have other means than direct calculation. Following the notation of G. B. Mathews' *Theory of Numbers*, we have (from the investigation given in Art. 196, p. 219)

$$j + k = \tfrac{1}{3}\left[-a' + \frac{2(n-1)}{3}\right] = 19$$

and $k = 10$ — being the number of pairs of consecutive numbers in (ii) and (iii) taken together as belonging to a single class.

$\therefore$ $j = 9$.

Since $b = j - k$, $b = -1$.

Substituting the values of A, B, C, a' and b in IX, we get the required sextic

$$x^6 + x^5 + 7x^4 + 63x^3 - 81x^2 - 353x + 541 = 0.$$

Similarly, we derive the corresponding sextic for the prime 97:

$$x^6 + x^5 - 40x^4 + 45x^3 + 236x^2 - 230x - 389 = 0.$$

Before concluding, it may be remarked that Pandit Oudh Upadhyaya's Sextics for the primes 31, 43, and 61 are wrong, each in one of the numerical co-efficients. The correct equations for these primes are respectively:—

(i) $x^6 + x^5 + 3x^4 + 11x^3 + 44x^2 + 36x + 32 = 0$

(ii) $x^6 + x^5 + 4x^4 - 23x^3 + 67x^2 - 50x + 44 = 0$

(iii) $x^6 + x^5 - 25x^4 + 8x^3 + 123x^2 - 126x + 27 = 0.$

I give below a table of values of A, B, C, a' and b for all primes of the form $6p + 1$, less than 100, corresponding to their least primitive roots.

Prime.	Least Primitive root.	A	B	C	a'	b
7	3	0	0	0	1	-1
13	2	0	0	1	-1	-1
19	2	0	0	1	3	-1
31	3	1	1	0	2	2
37	2	0	1	2	-3	1
43	3	0	2	1	-2	2
61	2	2	1	2	1	-3
67	2	2	2	1	-1	3
73	5	2	1	3	3	-3
79	3	2	3	1	-5	-1
97	5	4	3	1	7	-1

A. A. KRISHNASWAMI IYENGAR.

Solutions.

Question 1309.

(S. N. KUMARASWAMY) :—In any triangle ABC, draw a straight line AD to the side BC such that the radii of the incircles of ABD, ACD may be equal. Explain any ambiguity in the solution.

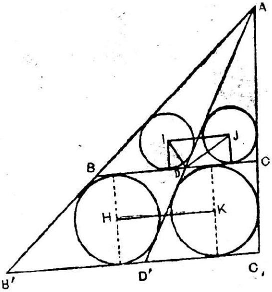
Solution (1) by 'Anon,' (2) by P. R. Venkatakrishna Iyer, (3) by I. Totadri Iyengar and S. Audinarayanan, (4) by K. J. Sanjana, N. M. Mehta and M. K. Kewalramani.

(1) Let the line AD be drawn so that the in-circles of ABD, ACD are equal. Then completing the figure, it is obvious that the ex-circles H, K are also equal, since they may be regarded as the in-circles of AB′D′, AC′D′, which are similar to ABD, ACD.

Now, let t_2, t_3 denote tangents from B, C to the circles I, J; also t'', t''' tangents from the same to the circles H, K. Then, if r, r' be the radii of the in- and ex-circles, from the right-angled triangle IDJ,

$$r^2 = t''.t''' \qquad \ldots \qquad \ldots \quad (1)$$

since the tangents from D to I and J are respectively t'' and t'''.



Similarly, from the right-angled triangle HDK,

$$r'^2 = t_2 . t_3. \quad \ldots \quad (2)$$

Further, since IBH is a right angle,

$$\frac{r}{t_2} = \frac{r'}{t''} = 1.$$

$$\therefore\ r.r' = t_2.t'' = t_3.t''' \ (3)$$

Again, if ADC = θ,

$$r = t'''.\tan\frac{\theta}{2};\ r = t''.\cot\frac{\theta}{2}$$

$$\therefore\quad t'' = t'''.\tan^2\frac{\theta}{2}.$$

$$\therefore\qquad t_3 = t_2.\tan^2\frac{\theta}{2} \text{ by (3)}$$

$$\therefore\qquad \cos\theta = \frac{(t_2 - t_3)}{(t_2 + t_3)} \qquad \ldots \qquad \ldots \quad (4)$$

$$= \frac{(t - t')}{IJ}$$

where t, t' are tangents from A to the circles I and J.

Thus, we have

$$\begin{aligned}(c-b) &= (t+t_2-t'-t_3)\\ &= (\mathrm{IJ}+t_2+t_3)\cos\theta\\ &= a\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad (5)\end{aligned}$$

In other words, the inclination of AD to BC is given by the formula

$$\cos\theta = \frac{(c-b)}{a}.$$

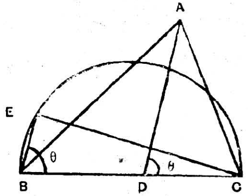
[*Note*: The Proposer gives the following construction for AD :—

With M the mid-point of BC as centre describe a circle of radius $\frac{1}{2}(b+c)$ to cut the altitude XA at E; then AD will be parallel to ME. The proof is, however, tedious.]

(2) On BC describe a semi-circle and with B as centre and radius the difference of AB and AC describe an arc cutting the semi-circle in E. Join BE and through A draw AD parallel to BE. Then AD is the required line.

Let ADC $= \theta$. Then, from the figure,

$$\cos\theta = \frac{\mathrm{BE}}{\mathrm{BC}} = \frac{c-b}{a} = \frac{\sin\frac{\mathrm{C}-\mathrm{B}}{2}}{\sin\frac{\mathrm{C}+\mathrm{B}}{2}} \quad \ldots \quad \ldots \quad (\text{i})$$



Again, the in-radius ρ_1 of triangle ABD

$$= \frac{\mathrm{AD}\sin\frac{\mathrm{BAD}}{2}\sin\frac{\mathrm{BDA}}{2}}{\cos\frac{\mathrm{B}}{2}} \quad \ldots \quad \ldots \quad (\text{ii})$$

and the in-radius ρ_2 of the triangle ACD

$$= \frac{\text{AD} \sin \frac{\text{CAD}}{2} \sin \frac{\text{ADC}}{2}}{\cos \frac{\text{C}}{2}} \quad \ldots \quad \ldots \text{(iii)}$$

If $\rho_1 = \rho_2$, $\quad \dfrac{\sin \frac{1}{2}(\theta - \text{B}) \cdot \cos \frac{1}{2}\theta}{\cos \frac{1}{2}(\theta + \text{C}) \cdot \sin \frac{1}{2}\theta} = \dfrac{\cos \frac{\text{B}}{2}}{\cos \frac{\text{C}}{2}}$,

i.e., $\quad \sin\left(-\frac{\text{B}}{2} + \theta\right) \cos \frac{\text{C}}{2} - \sin\left(\frac{\text{C}}{2} + \theta\right) \cos \frac{\text{B}}{2} = \sin \frac{\text{B} - \text{C}}{2}$.

Simplifying

$$\cos \theta . \sin \frac{\text{B} + \text{C}}{2} = \sin \frac{\text{C} - \text{B}}{2}$$

which is identical with (i).

(3) Let AD $= y$ and BD $= x$, so that DC $= a - x$.

Since the in-radii of the two triangles ABD and ACD are equal,

$$\frac{\Delta \text{ ABD}}{\Delta \text{ ACD}} = \frac{x + y + c}{y + b + a - x} = \frac{\text{BD}}{\text{CD}}.$$

i.e.,
$$\frac{x + y + c}{y + a + b - x} = \frac{x}{a - x},$$

On cross-multiplication, we have

$$y(a - 2x) = (b + c)x - ac.$$

Now
$$a y^2 = a\{c^2 + x^2 - 2cx \cos \text{B}\}$$
$$= a(c^2 + x^2) - x(c^2 + a^2 - b^2).$$

Eliminating y between these two, we get the biquadratic

$$\{a(c^2 + x^2) - x(c^2 + a^2 - b^2)\}\{4x^2 - 4ax + a^2\}$$
$$= a\{(b + c)^2 x^2 - 2ac(b + c)x + a^2c^2\}.$$

On simplifying and removing the two obvious roots $x = 0$ and $x = a$, we get the quadratic

$$4ax^2 - 4x(c^2 + a^2 - b^2) + a\{(a^2 + 4c^2) - (b + c)^2\} = 0; \quad \text{(A)}$$

Hence in general there are two solutions corresponding to

$$\text{AD} = x = \frac{(c^2 + a^2 - b^2) \pm (b - c)\sqrt{(b + c)^2 - a^2}}{2a}.$$

Since $b + c > a$, the roots are always real, and therefore the construction is always possible.

(i) If $\hat{B}$ *is acute*, $c^2 + a^2 > b^2$; if further $4c^2 + a^2 > (b + c)^2$ we see from (A) that there are two solutions: else if $4c^2 + a^2 \overline{\overline{<}} (b+c)^2$ there is only one solution.

(ii) If $\hat{B}$ *is right*, $c^2 + a^2 = b^2$. There is only one solution given by

$$x = \frac{1}{2a}(b - c)\sqrt{(b + c)^2 - a^2}.$$

(iii) If $\hat{B}$ *is obtuse*, $b^2 > c^2 + a^2$ and $b > c$.

$\therefore \quad b^2 + 2bc + c^2 > c^2 + a^2 + 2c^2 + c^2$, *i.e.* $(b + c)^2 > 4c^2 + a^2$.

There is again only one solution from (A).

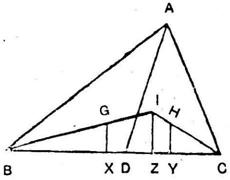
Hence there is ambiguity only when the two base angles are acute and the further relation in (i) is satisfied.

(4) Bisect the angles ABC and ACD by straight lines meeting in I the incentre; let G, H be the incentres of the triangles ABD, ADC in BI and CI respectively, and draw perpendiculars GX, HY, IZ to BC Let ρ be the radius of each of the equal in-circles and AD $= l$, BD $= x$, CD $= y$.

As ρ is the in-radius of ABD as well as of ACD, we have

$$\rho = \frac{2\,\Delta ABD}{c + x + l} = \frac{2\,\Delta ACD}{b + y + l}.$$

$$\therefore \quad \rho = \frac{2\,\Delta ABC}{a + b + c + 2l} = \frac{rs}{s + l}.$$



$$\therefore \quad \frac{\rho}{r} = \frac{s}{s + l}. \quad \ldots \quad \ldots \quad (1)$$

As the pairs of Δs BGX, BIZ and CHY, CIZ are similar, we have

$$\frac{\rho}{r} = \frac{BX}{BZ} = \frac{CY}{CZ};$$

$$\therefore \quad \frac{\rho}{r} = \frac{BX + CY}{BC}.$$

$$\therefore \quad \frac{\rho}{r} = \frac{\frac{1}{2}(c+x-l) + \frac{1}{2}(b+y-l)}{a} = \frac{s-l}{a}. \quad \dots \quad (2)$$

From (1) and (2) we have $s^2 - l^2 = a\,s$, so that $l = \sqrt{s\,(s-a)}$.

Thus the point D is determined as the intersection of BC by a circle whose centre is A and radius is $\sqrt{s\,(s-a)}$.

Note.—When the ex-circles of ABD, ACD (each of radius ρ_1) opposite to the angles at A are equal, it is found by a similar process that equations (1) and (2) are replaced by

$$\frac{\rho_1}{r_1} = \frac{s-a}{l+(s-a)} = \frac{l-(s-a)}{a};$$

whence $$l^2 - (s-a)^2 = a\,(s-a),$$

$$\therefore \quad l^2 = s\,(s-a),$$

so that the same length for AD is determined as before.

Supposing AB and AC given and AB > AC, we shall have

$$\sqrt{s\,(s-a)} = b$$

when $a^2 = (c-b)(c+3b)$, or $\cos A = (2b-c)/c$.

In this case, the second point of intersection coincides with C and there is only one geometrical position of AD.

When $$a < \sqrt{(c-b)(c+3b)},$$

or $$\cos A < (2b-c)/a,$$

the second point of intersection falls outside the triangle and geometrically there is only one solution. But when a has a greater value, both points of section fall between B and C; in this case, it is not difficult to see that the position of AD nearer to the greater side AB makes the in-circles (or ex-circles) equal, but the actual analytical proof is troublesome.

Questions for Solution.

1390. (S. D. Chowla):—Prove that

$$\sum_{n=1}^{\infty}\left[\frac{e^{-n\pi^2}}{(1-e^{-2\pi})(1-e^{-4\pi})\ldots\ldots(1-e^{-2\pi n})}\right]^2 = \frac{\sqrt{2}\,\Gamma(\tfrac{3}{4})e^{-\pi/12}-\pi^{\frac{1}{4}}}{\pi^{\frac{1}{4}}}.$$

1391. (V. Tiruvenkatacharya):—Show that

(i) $$\sum_{n=0}^{\infty}\frac{\cos(6n+4)\dfrac{\theta}{3}}{\dfrac{6n+4}{3}} = \frac{1}{4}\log\frac{\sin\theta}{4\sin^3\dfrac{\theta}{3}} - \frac{\pi}{4\sqrt{3}};$$

(ii) $$\sum_{n=0}^{\infty}\frac{\sin(6n+4)\dfrac{\theta}{3}}{\dfrac{6n+4}{3}} = \frac{3}{4}\log\frac{2\left(\sqrt{3}-\tan\dfrac{\theta}{3}\right)}{2\left(\sqrt{3}+\tan\dfrac{\theta}{3}\right)} + \frac{\pi}{4} - \frac{\sqrt{3}}{4}\log 2$$

where $0 < \theta < \pi$.

1392. (S. N. Kumaraswami):—Shew how to draw n straight lines from the vertex A to the base BC of a triangle ABC, such that the radii of the in-circles of the $(n+1)$ triangles so formed may be equal. Discuss whether the radii of the ex-circles of the triangles are also equal.

1393. (R. Gopalaswami):—Shew that a conic touching the sides of a triangle ABC at L, M, N has double contact with the polar conic of the centre of perspective of the triangles ABC and LMN and that the chord of contact is their axis of perspective.

Deduce that the Brocard ellipse has double contact with the circum-circle

1394. (S. Narayana Aiyar):—Show that

(1) $$\prod_{r=1}^{n-1}\operatorname{dn}\frac{rK}{n} = \prod_{r=1}^{n-1}\operatorname{cs}\frac{rK}{n} = k')^{\frac{n-1}{2}}.$$

(2) $$\prod_{r=1}^{n-1}\operatorname{cn}\frac{r}{n}(K+iK') = \{\operatorname{cn}(K+iK')\}^{\frac{n-1}{2}}.$$

(3) $$\prod_{r=1}^{n-1}\operatorname{ds}\frac{r}{n}(K+iK') = k^{n-1}\{\operatorname{cn}(K+iK')\}^{\frac{n-1}{2}}.$$

where n is a positive integer and r takes all the values from 1 to $n-1$ in the product

1395. (R. GOPALASWAMI) :—Given a quadrilateral and a conic, it is required to draw a conic osculating the given conic and having for conjugate points the 3 pairs of opposite vertices of the quadrilateral.

Show that there are four distinct conics satisfying the conditions.

1396. (V. RAMASWAMI AIYAR):—Given four concyclic points $(O_1\ O_2\ O_3\ O_4)$, if the orthocentres of the triangles whose vertices they are, be denoted by $(H_1\ H_2\ H_3\ H_4)$ in the appropriate manner, show that the set of points $(H_1\ H_2\ H_3\ H_4)$ as well as the six sets of points like $(O_1\ O_2\ H_3\ H_4)$ lie on circles equal to the circle $(O_1\ O_2\ O_3\ O_4)$.

1397. (MARTYN THOMAS):—If

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ \ldots$$

be a convergent recurring series of the third order, show that its sum to infinity is

$$\begin{vmatrix} 0 & s_1 x^2 & s_2 x & s_3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} \div \begin{vmatrix} x^3 & x^2 & x & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix}$$

where s_1, s_2, s_3 denote the sum of the first 1, 2, 3 terms of the series, respectively.

1398. (MARTYN THOMAS) :—Show that the singular solution of the differential equation

$$4\left(x + y\frac{dy}{dx}\right)^2 + 25\left(y^2 - xy\frac{dy}{dx}\right) = 0$$

is the envelope of a series of circles described on the sub-normal of an ellipse as diameter.

1399. (A. NARASINGA RAO, M.A.):— $F(x, y) = \Sigma a_{rs} x^r y^s$ is a symmetric polynomial of degee n in the two variables x and y separately and is such that any three of the relations

$$F(x_1 y_1) = 0 \quad F(x_2 y_1) = 0$$
$$F(x_1 y_2) = 0 \quad F(x_2 y_2) = 0$$

imply the fourth.

Show that F may be reduced to the canonical form

$$F(x_1 y) \equiv f(x) f(y) + g(x) g(y)$$

where f and g are polynomials of degree n.

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LIST OF JOURNALS RECEIVED IN THE LIBRARY.

(during the month of June 1925).

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1 Acta Mathematica, Vol. 45, Nos. 3 & 4.

2 Annales l'Observatoire de Paris, February 1925.

3 Astrophysical Journal, Vol. 61, No. 3, April 1925.

4 Japanese Journal of Mathematics, Vol. 1, No. 3.

5 Mathematische Annalen 946 and 1/2 heft.

6 Mathematical Gazette, Vol. 12, No. 174, May 1925. 2 copies.

7 Messenger of Mathematics, Vol. 54, No. 10, February 1925. 2 copies.

8 Monthly Notices of the Royal Astronomical Society, Vol. 85, No. 5, March 1925.

9 Nature, Vol. 115, Nos. 2896, 2897, 2898.

10 Philosophical Magazine & Journal of Sciences [The London, Edinburgh and Dublin], Vol. 49, No. 294, June 1925.

11 Popular Astronomy, Vol. 33, No. 5.

12 Proceedings of the London Mathematical Society, Vol. 24, Parts 1 and 2. 3 copies of each Part.

13 Proceedings of the Royal Society of London, Vol. 108, No. A 745, May 1925.

14 The Tohoku Mathematical Journal, Vol. 24, Nos. 3 and 4.

15 Transactions of the American Mathematical Society, Vol. 27, No. 2.

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